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On an anisotropic Serrin criterion for weak solutions of the Navier–Stokes equations

Sur un critère de Serrin anisotrope pour des solutions faibles de l'équation de Navier–Stokes

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ABSTRACT

In this paper, we draw on the ideas of [5] to extend the standard Serrin criterion [18] to an anisotropic version thereof. Because we work on weak solutions instead of strong ones, the functions involved have low regularity. Our method summarizes in a joint use of a uniqueness lemma in low regularity and the existence of stronger solutions. The uniqueness part uses duality in a way quite similar to the DiPerna–Lions theory, first developed in [7]. The existence part relies on L^p energy estimates, whose proof may be found in [5], along with an approximation procedure.

RÉSUMÉ

Dans cet article, nous utilisons les idées de [5] pour étendre le critère usuel de Serrin [18] à un cas anisotrope. Puisque nous nous intéressons à des solutiosn faibles et non fortes des équations de Navier–Stokes, les fonctions en jeu possèdent une faible régularité. Notre méthode peut se résumer à l'usage conjoint d'un lemme d'unicité en basse régularité avec un lemme d'existence de solutions plus régulières. La partie concernant l'unicité fait usage de la dualité d'une manière rappelant la théorie de Diperna–Lions, exposée pour la première fois dans [7]. La partie concernant l'existence repose sur des estimations d'énergie dans L^p, dont la preuve se trouve dans [5], ainsi que sur une procédure d'approximation standard.

1. Presentation of the problem

The present paper deals with the regularity of the Leray solutions of the incompressible Navier–Stokes equations in dimension three in space. We recall that these equations are

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$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) - \Delta u = -\nabla p, & t \geq 0,\ x \in \mathbb{X}^3, \\ \operatorname{div} u \equiv 0, \\ u(0) = u_0. \end{cases} \tag{1}$$

Here, $u = (u^1, u^2, u^3)$ stands for the velocity field of the fluid, p is the pressure and we have set for simplicity the viscosity equal to 1. We use the letter $\mathbb{X}$ to denote $\mathbb{R}$ and $\mathbb{T}$ whenever the current claim or proposition applies to both of them. Let us first recall the existence theorem proved by J. Leray in his celebrated paper [13].

Theorem 1 *(J. Leray, 1934). Let us assume that u_0 belongs to the energy space $L^2(\mathbb{X}^3)$. Then there exists at least one vector field u in the energy space $L^\infty(\mathbb{R}_+, L^2(\mathbb{X}^3)) \cap L^2(\mathbb{R}_+, H^1(\mathbb{X}^3))$ which solves the system (1) in the weak sense. Moreover, the solution u satisfies for all $t \geq 0$ the energy inequality*

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{X}^3)} + \int_0^t \|\nabla u(s)\|^2_{L^2(\mathbb{X}^3)} ds \leq \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{X}^3)}.$$

Uniqueness of such solutions, however, remains an outstanding open problem to this day. In his paper from 1961 [18], J. Serrin proved that, if one assumes that there exists a weak solution which is mildly regular, then it is actually smooth in space. More precisely, J. Serrin proved that if a weak solution u belongs to $L^p(]T_1, T_2[, L^q(D))$ for $T_2 > T_1 > 0$ and some bounded domain $D \Subset \mathbb{X}$ with the restriction $\frac{2}{p} + \frac{3}{q} < 1$, then this weak solution is $\mathcal{C}^\infty$ in the space variable on $]T_1, T_2[\times D$. Following his path, many other authors proved results in the same spirit, with different regularity assumptions and/or covering limit cases. Let us cite for instance [3], [4], [5], [8], [9], [10], [11], [19], [20] and references therein. Closer to our paper is the result of J. Neustupa and P. Penel in [17]; their paper is the first about one-component regularity of weak solutions of the Navier–Stokes equations, though its main assumption is not scaling invariant.

In this paper, we prove two results of the type we mentioned above: the first one is stated in the torus, while the second one is in a spatial domain in the usual Euclidean space. Thanks to the compactness of the torus, the first result is easier to prove than its local-in-space counterpart. For this reason, we will use the torus case as a toy model, thus avoiding many technicalities and enlightening the overall strategy of the proof.

In the torus, the theorem writes as follows.

Theorem 2. *Let u be a Leray solution of the Navier–Stokes equations set in $\mathbb{R}_+ \times \mathbb{T}^3$*

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) - \Delta u = -\nabla p \\ u(0) = u_0 \end{cases}$$

with initial data u_0 in $L^2(\mathbb{T}^3)$ and assume that there exists a time interval $]T_1, T_2[$ such that its third component u^3 satisfies

$$u^3 \in L^2(]T_1, T_2[, W^{2,\frac{3}{2}}(\mathbb{T}^3)).$$

Then u is actually smooth in space on $]T_1, T_2[\times \mathbb{T}^3$.

In a subdomain of the whole space, we need to add a technical assumption on the initial data, namely that it belongs to some particular L^p space with $p < 2$. Notice that such an assumption is automatically satisfied in the torus, thank to its compactness.

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Theorem 3. *Let u be a Leray solution of the Navier–Stokes equations set in $\mathbb{R}_+ \times \mathbb{R}^3$*

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) - \Delta u = -\nabla p \\ u(0) = u_0 \end{cases}$$

with initial data u_0 in $L^2(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3)$ and assume that there exists a time interval $]T_1, T_2[$ and a spatial domain $D \Subset \mathbb{R}^3$ of compact closure such that its third component u^3 satisfies

$$u^3 \in L^2(]T_1, T_2[, W^{2,\frac{3}{2}}(D)).$$

Then, on $]T_1, T_2[\times D$, u is actually smooth in space.

Compared to the classical case, our result may seem weaker, as we require two space derivatives in $L^{\frac{3}{2}}$. However, the space in which we assume to have u^3 is actually at the same scaling that $L^2(]T_1, T_2[, L^\infty(D))$ or $L^2(]T_1, T_2[, BMO(D))$, which are more classically found in regularity theorems such as the one of J. Serrin. In the scaling sense, our assumption is as strong as the usual Serrin criterion. We demand a bit more in terms of spatial regularity to counterbalance the anisotropic nature of the criterion.

2. Overview of the proof

Our strategy draws its inspiration from the anisotropic rewriting of the Navier–Stokes system done in [5], though it also bears resemblance to the work of [1], [2], [6], [12], [14]. Letting

$$\Omega := \operatorname{rot} u = (\omega_1, \omega_2, \omega_3), \quad \omega := \omega_3,$$

we notice that ω solves a transport-diffusion equation with $\Omega \cdot \nabla u^3$ as a forcing term. This equation writes

$$\begin{cases} \partial_t \omega + \nabla \cdot (\omega u) - \Delta \omega = \Omega \cdot \nabla u^3 \\ \omega(0) = \omega_0, \end{cases} \tag{2}$$

for some ω_0 which we do not specify. Actually, because we will assume more regularity on u^3 than given by the J. Leray theorem on a time interval which does not contain 0 in its closure, we will focus our attention on a truncated version of ω, for which the initial data is equal to 0. For the clarity of the discussion to follow, we drop any mention of the cut-off terms in this section. In the same vein, we will act as if Lebesgue spaces on $\mathbb{R}^3$ were ordered, which is of course only true on compact subdomains of $\mathbb{R}^3$.

Viewing Equation (2) as some abstract PDE problem, we are able to show, by a classical approximation procedure, the existence of *some* solution, call it $\tilde{\omega}$, which belongs to what we shall call the energy space associated to $L^{\frac{6}{5}}(\mathbb{X}^3)$, namely

$$L^\infty(\mathbb{R}_+, L^{\frac{6}{5}}(\mathbb{X}^3)) \cap L^2(\mathbb{R}_+, \dot{W}^{1,\frac{6}{5}}(\mathbb{X}^3)).$$

Thanks to Sobolev embeddings, we have $L^{\frac{6}{5}}(\mathbb{X}^3) \hookrightarrow \dot{H}^{-1}(\mathbb{X}^3)$ and $\dot{W}^{1,\frac{6}{5}}(\mathbb{X}^3) \hookrightarrow L^2(\mathbb{X}^3)$. In particular, this energy space is a subspace of $L^2(\mathbb{R}_+ \times \mathbb{X}^3)$. We then conclude that $\tilde{\omega}$ is actually equal to ω thanks to a uniqueness result in $L^2(\mathbb{R}_+ \times \mathbb{X}^3)$ for Equation (2). In particular, our ω has now an improved regularity, a fact which we will prove useful in the sequel.

At this stage, two things are to be emphasized. The first one is that the uniqueness result comes alone, without any existential counterpart. To put it plainly, we are *not* able to prove existence of solutions in the class where we are seeking uniqueness, contrary to, for instance, the now classical results from DiPerna–Lions et al. The existence here is given from the outside by the very properties of the Navier–Stokes equations.

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The second one is the absence of any L^p bound uniform in time in the uniqueness class. From the algebra of the equation and the regularity assumption we made, one could indeed deduce boundedness in time but only in a Sobolev space of strongly negative index, like $H^{-2}(\mathbb{X}^3)$. The author is unaware of any uniqueness result for similar equations in such low-regularity spaces of distributions.

We then proceed to decompose the full vorticity Ω only in terms of ω and $\partial_3 u^3$, thanks to the div-curl decomposition, otherwise known as the Biot–Savart law. This decomposition essentially relies on the fact that a $2D$ vector field is determined by its $2D$ vorticity and divergence. In the case of (u^1, u^2), its $2D$ divergence is $-\partial_3 u^3$, because u is divergence free and its $2D$ vorticity is exactly ω.

Let us introduce some piece of notation, which is taken from [5]. We denote

$$\nabla_h := (\partial_1, \partial_2)\,,\ \nabla_h^\perp := (-\partial_2, \partial_1)\,,\ \Delta_h := \partial_1^2 + \partial_2^2.$$

Hence, we can write, denoting $u^h := (u^1, u^2)$,

$$u^h = u^h_{\mathrm{curl}} + u^h_{\mathrm{div}},$$

where

$$u^h_{\mathrm{curl}} := \nabla_h^\perp \Delta_h^{-1}\omega\,,\ u^h_{\mathrm{div}} := \nabla_h \Delta_h^{-1}(-\partial_3 u^3).$$

We thus obtain a decomposition of the force $\Omega \cdot \nabla u^3$ into a sum a terms which are of two types. The first are linear in both ω and u^3, while the others are quadratic in u^3 and contain no occurrence of ω. The first ones write as

$$\omega\partial_3 u^3 + \partial_2 u^3 \partial_3 u^1_{\mathrm{curl}} - \partial_1 u^3 \partial_3 u^2_{\mathrm{curl}},$$

while the terms quadratic in u^3 are

$$\partial_2 u^3 \partial_3 u^1_{\mathrm{div}} - \partial_1 u^3 \partial_3 u^2_{\mathrm{div}}.$$

In other words, our ω is now the solution of some modified, anisotropic transport-diffusion equation with forcing terms. The forcing terms are exactly those quadratic in u^3 mentioned above and by our assumption on u^3, they lie in $L^1(\mathbb{R}_+, L^{\frac{3}{2}}(\mathbb{X}^3))$.

We use again our strategy based on uniqueness. On this new, anisotropic equation, we prove a uniqueness result *in a regularity class in which ω now lies*, that is, in

$$L^\infty(\mathbb{R}_+, L^{\frac{6}{5}}(\mathbb{X}^3)) \cap L^2(\mathbb{R}_+, \dot{W}^{1,\frac{6}{5}}(\mathbb{X}^3)),$$

which is a space of functions more regular than the mere $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$ given by J. Leray existence theorem. We then proceed to prove the existence of a solution to this anisotropic equation in the energy space associated to $L^{\frac{3}{2}}(\mathbb{R}^3)$, which is

$$L^\infty(\mathbb{R}_+, L^{\frac{3}{2}}(\mathbb{X}^3)) \cap L^2(\mathbb{R}_+, W^{1,\frac{3}{2}}(\mathbb{X}^3)).$$

Again, Sobolev and Lebesgue embeddings (see the remark in the beginning of this section) entail that the energy space associated to $L^{\frac{3}{2}}(\mathbb{X}^3)$ embeds in that associated to $L^{\frac{6}{5}}(\mathbb{X}^3)$. Thanks to the second uniqueness result, we deduce once again that ω has more regularity than assumed. More precisely, we have proved that ω lies in

$$L^\infty(\mathbb{R}_+, L^{\frac{3}{2}}(\mathbb{X}^3)) \cap L^2(\mathbb{R}_+, W^{1,\frac{3}{2}}(\mathbb{X}^3)).$$

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Now that we have lifted the regularity of $\omega = \omega_3$ to that of ∇u^3, it remains to improve the two other components of the vorticity. Keeping in mind that we now control two independant quantities in a high regularity space instead of one as we originally assumed, the remainder of the proof shall be easier than its beginning.

At first sight, ω_1 and ω_2 solve two equations which both look very similar to Equation (2). Indeed, we have

$$\begin{cases} \partial_t\omega_1 + \nabla\cdot(\omega_1 u) - \Delta\omega_1 = \Omega\cdot\nabla u^1 \\ \partial_t\omega_2 + \nabla\cdot(\omega_2 u) - \Delta\omega_2 = \Omega\cdot\nabla u^2. \end{cases} \tag{3}$$

We again make use of the div-curl decomposition, but in a somewhat *adaptative* manner. Recall that, when we improved the regularity of ω_3, we performed a div-curl decomposition with respect to the third variable. Such a decomposition has the drawback of forcing the appearance of anisotropic operators, which make lose regularity in some variables and gain regularity in others.

Let us pause for a moment to notice something interesting. From the div-curl decomposition with respect to the third variable, we know that me way write

$$u^h := (u^1, u^2) = \nabla_h^\perp\Delta_h^{-1}\omega + \nabla_h\Delta_h^{-1}(-\partial_3 u^3).$$

Taking the horizontal gradient then gives

$$\nabla_h u^h = \nabla_h\nabla_h^\perp\Delta_h^{-1}\omega + \nabla_h^2\Delta_h^{-1}(-\partial_3 u^3).$$

That is, $\nabla_h u^h$ may be written as a linear combination of zero order *isotropic* differential operators applied to $\omega = \omega_3$ and $\partial_3 u^3$. In other words, as a consequence of the Hörmander–Mikhlin theorem in three dimensions, the four components of the Jacobian matrix $\partial_i u^j, 1 \le i, j \le 2$ have the same regularity as ω_3 and $\partial_3 u^3$.

Now that we have some regularity on both u^3 and ω_3, we may choose to perform the div-curl decomposition with respect to the second variable for u^1 and to the first variable for u^2. Since the $2D$ divergence of (u^3, u^1) is $-\partial_2 u^2$ and its $2D$ vorticity is ω_2, we have

$$u^1 = \partial_3\Delta_{(1,3)}^{-1}\omega_2 - \partial_1\Delta_{(1,3)}^{-1}\partial_2 u^2.$$

In turn, taking the derivative with respect to the third variable gives

$$\partial_3 u^1 = \partial_3^2\Delta_{(1,3)}^{-1}\omega_2 - \partial_3\partial_1\Delta_{(1,3)}^{-1}\partial_2 u^2.$$

That is, $\partial_3 u^1$ may be expressed as the sum of a term linear in ω_2 and a source term which is, for instance, in $L^2(\mathbb{R}_+, L^3(\mathbb{X}^3))$. A similar decomposition also applies to $\partial_3 u^2$. Consequently, the system on (ω_1, ω_2) may be recast informally in the following form

$$\begin{cases} \partial_t\omega_1 + \nabla\cdot(\omega_1 u) - \Delta\omega_1 = (\text{lin. term in } \omega_2) + (\text{source terms in } L^1(\mathbb{R}_+, L^{\frac{3}{2}}(\mathbb{X}^3))) \\ \partial_t\omega_2 + \nabla\cdot(\omega_2 u) - \Delta\omega_2 = (\text{lin. term in } \omega_1) + (\text{source terms in } L^1(\mathbb{R}_+, L^{\frac{3}{2}}(\mathbb{X}^3))). \end{cases}$$

Thus, it only remains to prove a uniqueness lemma similar to what we did for Equation (2), along with an existence statement in the energy space associated to $L^{\frac{3}{2}}(\mathbb{X}^3)$. We will then have proved that the full vorticity Ω was actually in, say, $L^4(\mathbb{R}_+, L^2(\mathbb{X}^3)))$, entailing that the whole velocity field lies in $L^1(\mathbb{R}_+, \dot{H}^1(\mathbb{X}^3)))$. A direct application of the standard Serrin criterion concludes the proof.

3. Notations

We define here the notations we shall use in this paper, along with some useful shorthands which we shall make a great use thereof.

If a is a real number or a scalar function, we define for $p > 0$ the generalized power a^p by

$$a^p := a|a|^{p-1}$$

if $a \neq 0$ and 0 otherwise. Such a definition has the advantage of being reversible, in that we have the equality $a = (a^p)^{\frac{1}{p}}$.

Spaces like $L^p(\mathbb{R}_t, L^q(\mathbb{X}^3_x))$ or $L^p(\mathbb{R}_t, W^{s,q}(\mathbb{X}^3_x))$ will have their name shortened simply to L^pL^q and $L^qW^{s,q}$.

As we will have to deal with anisotropy, spaces such as $L^p(\mathbb{R}_t, L^q(\mathbb{X}_z, L^r(\mathbb{X}^2_{x,y})))$ shall be simply written $L^pL^qL^r$ when the context prevents any ambiguity.

When dealing with regularizations procedures, often done through convolutions, we will denote the smoothing parameter by δ and the mollifying kernels by $(\rho_\delta)_\delta$.

If X is either a vector or scalar field which we want to regularize, we denote by X^δ the convolution $\rho_\delta * X$.

Conversely, assume that we have some scalar or vector field Y which is a solution of some (partial) differential equation whose coefficients are generically denoted by X. Both X and Y are to be thought as having low regularity. We denote by Y_δ the unique smooth solution of the same (partial) differential equation where all the coefficients X are replaced by their regularized counterparts X^δ.

If $1 \leq k \leq n$, the horizontal variable associated to the vertical variable k in $\mathbb{R}^n$ is the $n-1$ tuple of variables $(1, \dots, k-1, k+1, \dots, n)$. In practice, we will restrict our attention to $n = 3$, in which case the horizontal variable associated to, say, 3 is none other than $(1, 2)$.

Now, for $1 \leq i, j, k \leq 3$, we denote by $A^k_{i,j}$ the operator $\partial_i\partial_j\Delta^{-1}_{h_k}$, with h_k being the horizontal variable associated to the vertical variable k. We divide these 18 operators into three subsets.

First, we say that $A^k_{i,j}$ is *isotropic* if we have both $i \neq k$ and $j \neq k$. This corresponds to the case where the two derivatives lost through the derivations are actually gained by the inverse Laplacian. Applying the Hörmander–Mikhlin multiplier theorem in two dimensions shows that these operators are bounded from $L^p(\mathbb{X}^3)$ to itself for any $1 < p < \infty$. There are 9 such operators.

The second class is that of the $A^k_{i,j}$ for which exactly one on the two indices i and j is equal to k while the other is not. We say that such operators are *weakly anisotropic*. Here, we lose one derivative in the vertical variable and gain one in the horizontal variable. There are 6 such operators.

The third and last class, which we will not have to deal with in this paper thanks to the peculiar algebraic structure of the equations, is formed by the three $A^k_{k,k} = \partial^2_k\Delta^{-1}_{h_k}$ for $1 \leq k \leq 3$. To keep a consistent terminology, we call them *strongly anisotropic*. The fact that we lose two derivatives in the vertical variable and gain two derivatives in the horizontal variable while working in dimension 3 should make this last family quite nontrivial to study.

If A and B are two linear operators, their commutator is defined by $[A, B] := AB - BA$. We emphasize that, when dealing with commutators, we do not distinguish between a smooth function and the multiplication operator by the said function.

4. Preliminary lemmas

We collect in this section various results, sometimes taken from other papers which we will use while proving the main theorems. We begin by an analogue of the usual energy estimate, whose proof may be found in [5] except it is performed in L^p with $p \neq 2$.

Lemma 1. *Let $1 < p < \infty$ and a_0 in L^p. Let f be in $L^1 L^p$ and v be a divergence-free vector field in $L^2 L^\infty$. Assume that a is a smooth solution of*

$$\begin{cases} \partial_t a + \nabla \cdot (a \otimes v) - \Delta a = f \\ a(0) = a_0. \end{cases}$$

Then, $|a|^{\frac{p}{2}}$ belongs to $L^\infty L^2 \cap L^2 H^1$ and we have the L^p energy equality

$$\frac{1}{p}\|a(t)\|_{L^p}^p + (p-1)\int_0^t \||a(s)|^{\frac{p-2}{2}} \nabla a(s)\|_{L^2}^2 ds = \frac{1}{p}\|a_0\|_{L^p}^p + \int_0^t \int_{\mathbb{R}^3} f(s,x)a(s,x)|a(s,x)|^{p-2} dx ds.$$

Our next lemma is, along with the energy estimate above, one of the cornerstones of our paper. Thanks to it, we are able to prove that the solutions of some PDEs are more regular than expected. It may be found in [15] and appear as a particular case of Theorem 2 in [16], to which we refer the reader for a detailed proof.

Lemma 2. *Let v be a fixed, divergence free vector field in $L^2 H^1$. Let $\nu \geq 0$ be a real constant. Let a be a $L^2_{loc} L^2$ solution of*

$$\begin{cases} \partial_t a + \nabla \cdot (a \otimes v) - \nu \Delta a = 0 \\ a(0) = 0. \end{cases}$$

Then $a \equiv 0$.

The following lemma has a somewhat probabilistic flavor to it.

Lemma 3. *Let $(a_\delta)_\delta$ be a sequence of bounded functions in $L^p L^q$, with $1 \leq p, q \leq \infty$. Let a be in $L^p L^q$ and assume that*

$$\begin{cases} a_\delta \rightharpoonup^* a \text{ in } L^p L^q \\ a_\delta \to a \text{ a.e.} \end{cases}$$

as δ goes to 0.

Then, for any $\alpha \in]0,1[$, $a_\delta^\alpha \rightharpoonup^* a^\alpha$ in $L^{\frac{p}{\alpha}} L^{\frac{q}{\alpha}}$.

Proof. Let us fix some α in $]0,1[$ and let $p' := (1 - \frac{\alpha}{p})^{-1}$, $q' := (1 - \frac{\alpha}{q})^{-1}$. Let g be a smooth function with compact support in space, which we denote by S. Let us remark that, from the assumptions we made, $a_\delta^\alpha \to a^\alpha$ almost everywhere. By Egorov's theorem, because $[0,T] \times S$ has finite Lebesgue measure, for any $\varepsilon > 0$, there exists a subset A_ε of $[0,T] \times S$ of Lebesgue measure at most ε such that

$$\|a_\delta^\alpha - a^\alpha\|_{L^\infty(A_\varepsilon^c)} \to 0 \text{ as } \delta \to 0,$$

where we use A_ε^c as a shorthand for $([0,T] \times S) \setminus A_\varepsilon$. Out of the bad set A_ε, we can simply write

$$\left| \int_0^T \int_S (a_\delta^\alpha - a^\alpha) g \mathbb{1}_{A_\varepsilon^c} dx dt \right| \leq \|a_\delta^\alpha - a^\alpha\|_{L^\infty(A_\varepsilon^c)} \|g\|_{L^1 L^1},$$

and this last quantity goes to 0 as δ goes to 0, for any fixed ε. Let $\mu_\varepsilon(t) := \int_S \mathbb{1}_\varepsilon(t,x)dx$. We notice that $\|\mu\|_{L^1} \le \varepsilon$, while $\|\mu\|_{L^\infty} \le C$ for some C independant of ε. By interpolation, this gives $\|\mu\|_{L^{p'}} \lesssim \varepsilon^{\frac{1}{p'}}$. On A_ε, we have

$$\left|\int_0^T \int_S a_\delta^\alpha \mathbb{1}_{A_\varepsilon} g dxdt\right| \le \int_0^T \|a_\delta^\alpha\|_{L^{\frac{q}{\alpha}}(S)} \|g\|_{L^\infty} \mu_\varepsilon^{\frac{1}{q'}} dt$$

$$\le \|a_\delta^\alpha\|_{L^{\frac{p}{\alpha}} L^{\frac{q}{\alpha}}} \|g\|_{L^\infty L^\infty} \|\mu\|^{\frac{1}{q'}}_{L^{\frac{p'}{q'}}}$$

$$\lesssim \varepsilon^{\frac{1}{p'}}.$$

Similarly,

$$\left|\int_0^T \int_S a^\alpha \mathbb{1}_{A_\varepsilon} g dxdt\right| \lesssim \varepsilon^{\frac{1}{p'}}.$$

Letting first δ then ε go to 0, thanks to the fact that p' is finite, we get the desired convergence. The case of a general g in $L^{p'}L^{q'}$ is handled by a standard approximation procedure, which is made possible by the finiteness of both p' and q'. □

Lemma 4. *Let F be in L^1L^1 spatially supported in the ball $B(0,R)$ for some $R > 0$. Let a be the unique tempered distribution solving*

$$\begin{cases} \partial_t a - \Delta a = F \\ a(0) = 0. \end{cases}$$

Then there exists a constant $C = C_R > 0$ such that, for $|x| > 2R$, we have

$$|a(t,x)| \le C_R \|F\|_{L^1L^1} |x|^{-3}. \tag{4}$$

Proof. Let us write explicitly the Duhamel formula for a. We have, thanks to the support assumption on F,

$$a(t,x) = \int_0^t \int_{B(0,R)} (2\pi(t-s))^{-\frac{3}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} F(s,y)dyds.$$

As the quantity $\tau^{-3/2}e^{-A^2/\tau}$ reaches its maximum for $\tau = \frac{2A^2}{3}$, we have

$$|a(t,x)| \lesssim \int_0^t \int_{B(0,R)} |x-y|^{-3}|F(s,y)|dyds.$$

If x lies far away from the support of F, for instance if $|x| > 2R$ in our case, we further have

$$|a(t,x)| \le C_R \int_0^t \int_{B(0,R)} |x|^{-3}|F(s,y)|dyds = C_R|x|^{-3}\|F\|_{L^1L^1}. \quad \square$$

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The following lemma is an easy exercise in functional analysis, whose proof will be skipped.

Lemma 5. *Let us define, for some fixed $R > 0$ and $p > 1$, the space*

$$\tilde{W}^{1,p}(\mathbb{R}^3) := \{u \in W^{1,p}(\mathbb{R}^3) \ s.t. \sup_{|x|>2R} |x|^3|u(x)| < \infty\}.$$

Then the embedding of $\tilde{W}^{1,p}$ into L^p is compact.

Lemmas 6 to 12 are mostly variations on the same regularization tool, useful to sequentially gain regularity orders for weak solutions. The main ideas remain the same: regularize the exterior vector fields, prove energy-type estimates uniform in the regularization parameter, pass to the limit and conclude using a uniqueness argument in the weaker regularity class. The necessity of these variants comes from the variety of subtypes of equations we have to deal with, depending on whether there are – or not – zero-order Fourier multipliers (isotropic or anisotropic) or exterior forces in the right-hand side. In order to lighten the burden of the reader, we choose to not write the details of all the proofs, focusing our efforts on what we believe are the most important ones. Whenever a proof does not sensibly differ from an already written counterpart, we only refer to the latter instead of copying *mutadis mutandis* the former.

The next lemma combines some of the previous ones and plays a key role in the paper. It allows us to gain regularity on the solutions to transport-diffusion equations for free.

Lemma 6. *Let v be a fixed, divergence free vector field in L^2H^1. Let $\frac{6}{5} < p \leq 2$. Let $F = (F_i)_i$ be in L^1L^p and assume that $a = (a_i)_i$ is a solution in L^2L^2 of*

$$\begin{cases} \partial_t a + \nabla \cdot (a \otimes v) - \Delta a = F \\ a(0) = 0. \end{cases}$$

Then a is actually in $L^\infty L^p \cap L^2W^{1,p}$ and moreover, its i-th component a_i satisfies the energy inequality

$$\frac{1}{p}\|a_i(t)\|_{L^p}^p + (p-1)\int_0^t \||a_i(s)|^{\frac{p-2}{2}} \nabla a_i(s)\|_{L^2}^2 ds \leq \int_0^t \int_{\mathbb{R}^3} a_i^{p-1}(s)F_i(s)dxds.$$

Proof. Before delving into the proof itself, we begin with a simplifying remark. As the equation on a_i simply writes

$$\partial_t a_i + \nabla \cdot (a_i v) - \Delta a_i = F_i,$$

the equations on the a_i are uncoupled, which allows us prove to prove the lemma only in the scalar case. Thus, we assume in the rest of the proof the a is actually a scalar function.

Let $(\rho_\delta)_\delta$ be a sequence of space–time mollifiers. Let a_δ be the unique solution of the Cauchy system

$$\begin{cases} \partial_t a_\delta + \nabla \cdot (a_\delta v^\delta) - \Delta a_\delta = F^\delta \\ a_\delta(0) = 0. \end{cases}$$

Performing an energy-type estimate in L^p, which is made possible thanks to Lemma 1, we get for all strictly positive t the equality

$$\frac{1}{p}\|a_\delta(t)\|_{L^p}^p + (p-1)\int_0^t \||a_\delta(s)|^{\frac{p-2}{2}} \nabla a_\delta(s)\|_{L^2}^2 ds = \int_0^t \int a_\delta^{p-1}(s)F^\delta(s)dxds$$

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In turn, it entails that

$$\|a_\delta(t)\|_{L^p} \leq p \int_0^t \|F^\delta(s)\|_{L^p} ds,$$

which finally gives

$$\frac{1}{p}\|a_\delta(t)\|_{L^p}^p + (p-1)\int_0^t \||a_\delta(s)|^{\frac{p-2}{2}}\nabla a_\delta(s)\|_{L^2}^2 ds \leq p^{p-2}\left(\int_0^t \|F^\delta(s)\|_{L^p} ds\right)^p.$$

From the definition of F^δ, we infer that

$$\frac{1}{p}\|a_\delta(t)\|_{L^p}^p + (p-1)\int_0^t \||a_\delta(s)|^{\frac{p-2}{2}}\nabla a_\delta(s)\|_{L^2}^2 ds \leq p^{p-2}\left(\int_0^t \|F(s)\|_{L^p} ds\right)^p,$$

where the last term is independent of δ. Because $p < 2$, we have a bound on a_δ in $L^\infty L^p \cap L^2 W^{1,p}$ uniform in δ, thanks to the identity $\nabla a = (\nabla a|a|^{\frac{p-2}{2}})|a|^{\frac{2-p}{2}}$.

We now take the limit $\delta \to 0$. First of all, because F^δ is nothing but a space–time mollification of F, we have

$$\|F^\delta - F\|_{L^1 L^p} \to 0 \text{ as } \delta \to 0.$$

Moreover, the weak-$*$ accumulation points of $(a_\delta)_\delta$ in $L^\infty L^p$ and $L^2 W^{1,p}$ respectively are, in particular, solutions of the problem

$$\begin{cases} \partial_t b + \nabla\cdot(bv) - \Delta b = F \\ b(0) = 0. \end{cases}$$

Because $p \geq \frac{6}{5}$, the space $W^{1,p}(\mathbb{R}^3)$ embeds into L^q for some $q \geq 2$. By Lemma 2, the only possible accumulation point is none other than a. Thus, as $\delta \to 0$,

$$\begin{cases} a_\delta \rightharpoonup^* a \text{ in } L^\infty L^p \\ a_\delta \rightharpoonup a \text{ in } L^2 W^{1,p}. \end{cases}$$

From Lemma 4, we also have

$$|a_\delta(t,x)| \lesssim |x|^{-3}$$

for large enough x, with constants independant of δ. Combining the bounds we have on the family $(a_\delta)_\delta$, we have shown that this family is bounded in $L^2_{loc}\tilde{W}^{1,p}$. On the other hand, the equation on a_δ may be rewritten as

$$\partial_t a_\delta = -\nabla\cdot(a_\delta \otimes v^\delta) + \Delta a_\delta + F^\delta$$

and the right-hand side is bounded in, say, $L^1_{loc}H^{-2}$, because $p \geq \frac{6}{5}$. By Aubin–Lions lemma, it follows that the family $(a_\delta)_\delta$ is strongly compact in, say, $L^2_{loc}L^p$. Furthermore, once again thanks to Lemma 2, it follows that a is the only strong accumulation point of $(a_\delta)_\delta$ in $L^2_{loc}L^p$. Thus,

$$a_\delta \to a \text{ in } L^2_{loc} L^p.$$

Thanks to this strong convergence, up to extracting a subsequence $(\delta_n)_n$, we have

$$a_{\delta_n} \to a \text{ a.e. as } n \to \infty.$$

We are now in position to apply Lemma 3 to the sequence $(a_{\delta_n})_n$. With $\alpha = \frac{p}{2}$, we have

$$a_{\delta_n}^{\frac{p}{2}} \rightharpoonup^* a^{\frac{p}{2}} \text{ in } L^\infty L^2 \text{ as } n \to \infty,$$

while $\alpha = p - 1$ leads to

$$a_{\delta_n}^{p-1} \rightharpoonup^* a^{p-1} \text{ in } L^\infty L^{\frac{p}{p-1}} \text{ as } n \to \infty.$$

Using the identity $\nabla(a^{\frac{p}{2}}) = \frac{p}{2} a^{\frac{p-2}{2}} \nabla a$ and the energy inequality, we have

$$\sup_{n \in \mathbb{N}} \int_0^t \|\nabla(a_{\delta_n}^{\frac{p}{2}})\|_{L^2}^2 ds < \infty.$$

Since $a_{\delta_n}^{\frac{p}{2}} \rightharpoonup^* a^{\frac{p}{2}}$ in $L^\infty L^2$ as $n \to \infty$, applying Fatou's lemma to $a^{\frac{p}{2}}$ shows that

$$\int_0^t \|\nabla(a^{\frac{p}{2}})\|_{L^2}^2 ds \leq \liminf_{n \to \infty} \int_0^t \|\nabla(a_{\delta_n}^{\frac{p}{2}})\|_{L^2}^2 ds < \infty.$$

Taking the limit in the energy inequality, we finally have

$$\frac{1}{p}\|a(t)\|_{L^p}^p + (p-1)\int_0^t \||a(s)|^{\frac{p-2}{2}} \nabla a(s)\|_{L^2}^2 ds \leq p^{p-2} \left(\int_0^t \|F(s)\|_{L^p} ds \right)^p.$$

More interestingly, taking the limit in the energy equality gives us the stronger statement

$$\frac{1}{p}\|a(t)\|_{L^p}^p + (p-1)\int_0^t \||a(s)|^{\frac{p-2}{2}} \nabla a(s)\|_{L^2}^2 ds \leq \int_0^t \int_{\mathbb{R}^3} a^{p-1}(s) F(s) dx ds.$$

The proof of the lemma is now complete. □

Lemma 7. *Let v be a fixed, divergence free vector field in $L^2 H^1$. Let A be a matrix-valued function in $L^2 L^3$. Let K be a matrix whose coefficients are homogeneous Fourier multipliers of order 0, smooth outside the origin. Let a be a solution in $(L^2 L^2)^2$ of the equation*

$$\begin{cases} \partial_t a + \nabla \cdot (a \otimes v) - \Delta a = AKa \\ a(0) = 0. \end{cases}$$

Then $a = 0$.

Proof. From the assumptions we made, the right-hand side AKa lies in $L^1L^{\frac{6}{5}}$. Thanks to Lemma 6, a is actually in $L^\infty L^{\frac{6}{5}} \cap L^2 W^{1,\frac{6}{5}}$. Moreover, we also have a set of energy estimates in $L^{\frac{6}{5}}$ on the components a_i of a, which are

$$\frac{5}{6}\|a_i(t)\|_{L^{\frac{6}{5}}}^{\frac{6}{5}} + \frac{1}{5}\int_0^t \||a_i(s)|^{-\frac{2}{5}}\nabla a_i(s)\|_{L^2}^2 ds \leq \int_0^t \int_{\mathbb{R}^3} a_i^{\frac{1}{5}}(s)(A(s)Ka(s))_i dxds.$$

By Hölder inequality and Sobolev embeddings, we have

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^3} a_i^{\frac{1}{5}}(s)(A(s)Ka(s))_i dxds &\lesssim \sum_j \int_0^t \|A(s)\|_{L^3}\|a_i(s)\|_{L^{\frac{6}{5}}}^{\frac{1}{5}}\|a_j(s)\|_{L^2} ds \\ &\lesssim \sum_j \int_0^t \|A(s)\|_{L^3}\|a_i(s)\|_{L^{\frac{6}{5}}}^{\frac{1}{5}}\|\nabla a_j(s)|a_j(s)|^{-\frac{2}{5}}\|_{L^2}\|a_j(s)\|_{L^{\frac{6}{5}}}^{\frac{2}{5}} ds \\ &\lesssim \sum_j \int_0^t \|A(s)\|_{L^3}\|a(s)\|_{L^{\frac{6}{5}}}^{\frac{3}{5}}\|\nabla a_j(s)|a_j(s)|^{-\frac{2}{5}}\|_{L^2} ds.\end{aligned}$$

Young inequality now ensures that

$$\int_0^t \|A(s)\|_{L^3}\|a(s)\|_{L^{\frac{6}{5}}}^{\frac{3}{5}}\|\nabla a_j(s)|a_j(s)|^{-\frac{2}{5}}\|_{L^2} ds \leq \frac{1}{10}\int_0^t \|\nabla a_j(s)|a_j(s)|^{-\frac{2}{5}}\|_{L^2}^2 ds + C\int_0^t \|A(s)\|_{L^3}^2\|a(s)\|_{L^{\frac{6}{5}}}^{\frac{6}{5}} ds.$$

Adding these inequalities and canceling out the gradient terms, we get

$$\frac{5}{6}\|a(t)\|_{L^{\frac{6}{5}}}^{\frac{6}{5}} \lesssim \int_0^t \|A(s)\|_{L^3}^2\|a(s)\|_{L^{\frac{6}{5}}}^{\frac{6}{5}} ds.$$

Grönwall inequality now implies that $a = 0$. □

Lemma 8. *Let $\frac{6}{5} \leq p \leq 2$. Let v be a fixed, divergence free vector field in L^2H^1. Let A be a matrix-valued function in L^2L^3. Let K be a matrix whose coefficients are homogeneous Fourier multipliers of order 0, smooth outside the origin. Let F be a fixed function in L^1L^p. Let a be a solution in $(L^2L^2)^2$ of the equation*

$$\begin{cases}\partial_t a + \nabla\cdot(a\otimes v) - \Delta a = AKa + F \\ a(0) = 0.\end{cases}$$

Then a is actually in $L^\infty L^p \cap L^2W^{1,p}$.

Proof. The proof follows closely the steps of Lemma 6, so we shall skip it. □

Lemma 9. *Let v be a fixed, divergence free vector field in L^2H^1. Let a be a $L^\infty L^{\frac{6}{5}} \cap L^2W^{1,\frac{6}{5}}$ solution of the linear system*

$$\begin{cases}\partial_t a + \nabla\cdot(av) - \Delta a = \alpha a + \sum_{i,j=1,2}\varepsilon_{i,j}(\partial_j\beta_i)A^3_{3,i}a \\ a(0) = 0,\end{cases} \tag{5}$$

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with $\varepsilon_{i,j} \in \{0,1\}$ *for any* $1 \leq i,j \leq 2$. *We also assume that* α *lies in* L^2L^3 *and that all the* β_i*'s are in* $L^2H^{\frac{3}{2}}$. *Then* $a \equiv 0$.

Proof. For the sake of readability, we assume in the proof that only one coefficient $\varepsilon_{i,j}$ is not zero. We denote the corresponding $\partial_j\beta_i$ simply by $\partial_j\beta$. Let us denote by F the right-hand side of (5). From the assumptions and anisotropic Sobolev embeddings, it follows that F belongs to $L^1L^{\frac{6}{5}}$. By Lemma 6, a satisfies an energy inequality which writes, in our case,

$$\frac{5}{6}\|a(t)\|^{\frac{6}{5}}_{L^{\frac{6}{5}}} + \frac{1}{5}\int_0^t \||a(s)|^{-\frac{2}{5}}\nabla a(s)\|^2_{L^2} ds \leq \int_0^t\int_{\mathbb{R}^3} \left(a^{\frac{6}{5}}(s)\alpha(s) + a^{\frac{1}{5}}(s)\partial_j\beta(s)A^3_{3,i}a(s)\right) dxds.$$

By Hölder inequalities, we have

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^3} a^{\frac{6}{5}}(s)\alpha(s)dxds &\lesssim \int_0^t \|a^{\frac{3}{5}}(s)\|^2_{L^3}\|\alpha(s)\|_{L^3}ds \\ &\lesssim \int_0^t \|a^{\frac{3}{5}}(s)\|_{L^2}\||a(s)|^{-\frac{2}{5}}\nabla a(s)\|_{L^2}\|\alpha(s)\|_{L^3}ds \\ &\leq \frac{1}{10}\int_0^t \||a(s)|^{-\frac{2}{5}}\nabla a(s)\|^2_{L^2}ds + C\int_0^t \|a^{\frac{3}{5}}(s)\|^2_{L^2}\|\alpha(s)\|^2_{L^3}ds.\end{aligned}$$

To bound the other term, we begin by using a trace theorem on β, which gives $\beta \in L^2L^\infty H^1$. Taking a horizontal derivative, we get $\partial_j\beta \in L^2L^\infty L^2$. We emphasize that such a trace embedding would not be true in general, because $H^{\frac{1}{2}}(\mathbb{X})$ does not embed in $L^\infty(\mathbb{X})$. Here, the fact that the multiplicator $\partial_j\beta$ appears as a derivative of some function is crucial. Regarding the weakly anisotropic term $A^3_{3,i}a$, the assumption on a gives $\partial_3 a \in L^2L^{\frac{6}{5}} = L^2L^{\frac{6}{5}}L^{\frac{6}{5}}$. Since in two dimensions the space $W^{1,\frac{6}{5}}$ embeds into L^3, we get $A^3_{3,i}a \in L^2L^{\frac{6}{5}}L^3$. Combining these embeddings with Hölder inequality, we arrive at

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^3} a^{\frac{1}{5}}(s)\partial_j\beta(s)A^3_{3,i}a(s)dxds &\leq \int_0^t \|a^{\frac{1}{5}}(s)\|_{L^6L^6}\|\partial_j\beta(s)\|_{L^\infty L^2}\|A^3_{3,i}a(s)\|_{L^{\frac{6}{5}}L^3}ds \\ &\lesssim \int_0^t \|a^{\frac{1}{5}}(s)\|_{L^6}\|\beta(s)\|_{H^{\frac{3}{2}}}\|\nabla a(s)\|_{L^{\frac{6}{5}}}.\end{aligned}$$

Using the identity $\nabla a = \left(|a|^{-\frac{2}{5}}\nabla a\right)|a|^{\frac{2}{5}}$ and Hölder inequality again, we get

$$\int_0^t\int_{\mathbb{R}^3} a^{\frac{1}{5}}(s)\partial_j\beta(s)A^3_{3,i}a(s)dxds \lesssim \int_0^t \|a^{\frac{3}{5}}(s)\|_{L^2}\|\beta(s)\|_{H^{\frac{3}{2}}}\||a(s)|^{-\frac{2}{5}}\nabla a(s)\|_{L^2}.$$

Now, Young inequality for real numbers entails, for some constant C,

$$\int_0^t\int_{\mathbb{R}^3} a^{\frac{1}{5}}(s)\partial_j\beta(s)A^3_{3,i}a(s)dxds \leq \frac{1}{10}\int_0^t \||a(s)|^{-\frac{2}{5}}\nabla a(s)\|^2_{L^2}ds + C\int_0^t \|a^{\frac{3}{5}}(s)\|^2_{L^2}\|\beta(s)\|^2_{H^{\frac{3}{2}}}ds.$$

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Canceling out the gradient terms, we finally get

$$\|a^{\frac{3}{5}}(t)\|_{L^2}^2 \lesssim \int_0^t \|a^{\frac{3}{5}}(s)\|_{L^2}^2(\|\alpha(s)\|_{L^3}^2 + \|\beta(s)\|_{H^{\frac{3}{2}}}^2)ds.$$

Grönwall's inequality then ensures that $\|a^{\frac{3}{5}}(t)\|_{L^2}^2 \equiv 0$ and thus that $a \equiv 0$. $\square$

The three following lemmas allow us, in the spirit of Lemmas 6 and 9, to enhance the regularity of the solutions to some equations. As their proofs are akin to those of the aforementioned Lemmad we only sketch them.

Lemma 10. *Let $\frac{6}{5} \leq p \leq 2$. Let v be a fixed, divergence free vector field in L^2H^1. Let a be a solution in $L^\infty L^{\frac{6}{5}} \cap L^2W^{1,\frac{6}{5}}$ of the linear system*

$$\begin{cases} \partial_t a + \nabla\cdot(av) - \Delta a = \alpha a + \sum\limits_{i,j=1,2} \varepsilon_{i,j}(\partial_j\beta_i)A_{3,i}^3 a + F \\ a(0) = 0, \end{cases}$$

with $\varepsilon_{i,j} \in \{0,1\}$ for any $1 \leq i,j \leq 2$. We also assume that α lies in L^2L^3, that all the β_i's are in $L^2H^{\frac{3}{2}}$ and that the force F belongs to $L^1L^p \cap L^1L^{\frac{6}{5}}$. Then a is actually in $L^\infty L^p \cap L^2W^{1,p}$.

Sketch of proof. For simplicity, we again assume that only one coefficient $\varepsilon_{i,j}$ is nonzero and write $\partial_j\beta$ instead of $\partial_j\beta_i$. We abbreviate the whole right-hand side of the equation by $\tilde{F}$. First, we mollify the force fields $\alpha, \partial_j\beta, F$ and the weakly anisotropic operator $A_{3,i}^3$ by some regularizing kernel ρ_δ. This mollified right-hand side will be denoted by $\tilde{F}^\delta$, even though it is not exactly equal to $\rho_\delta * \tilde{F}$. This regularization allows us to build smooth solutions a_δ to the modified equation. In a second step, Lemma 1 gives us estimates in the energy space associated to L^p which are uniform in δ. These estimates write, recalling that $a_\delta(0) = 0$,

$$\frac{1}{p}\|a_\delta(t)\|_{L^p}^p + (p-1)\int_0^t \||a_\delta(s)|^{\frac{p-2}{2}}\nabla a_\delta(s)\|_{L^2}^2 ds = \int_0^t\int_{\mathbb{R}^3} a_\delta(s,x)^{p-1}\tilde{F}^\delta(s,x)dxds.$$

Repeating the computations we did for Lemma 9 and using Hölder inequality to deal with F^δ, we get

$$\|a_\delta(t)\|_{L^p}^p \lesssim \int_0^t \|a_\delta(s)\|_{L^p}^p(\|\alpha^\delta(s)\|_{L^3}^2 + \|\beta^\delta(s)\|_{H^{\frac{3}{2}}}^2)ds + \int_0^t \|a_\delta(s)\|_{L^p}^{p-1}\|F^\delta(s)\|_{L^p}ds.$$

We detail here how to deal with the new term added by F^δ. Let us denote, for $T > 0$,

$$M_\delta(T) := \sup_{0\leq t\leq T}\|a_\delta(t)\|_{L^p}.$$

For $0 \leq t \leq T$, we have

$$\begin{aligned} \|a_\delta(t)\|_{L^p}^p &\lesssim \int_0^T \|a_\delta(s)\|_{L^p}^p(\|\alpha^\delta(s)\|_{L^3}^2 + \|\beta^\delta(s)\|_{H^{\frac{3}{2}}}^2)ds + M_\delta(T)^{p-1}\int_0^T \|F^\delta(s)\|_{L^p}ds \\ &\lesssim \int_0^T \|a_\delta(s)\|_{L^p}^p(\|\alpha^\delta(s)\|_{L^3}^2 + \|\beta^\delta(s)\|_{H^{\frac{3}{2}}}^2)ds + M_\delta(T)^{p-1}\|F\|_{L^1L^p}. \end{aligned}$$

Taking the supremum over $0 \le t \le T$ in the left-hand side gives

$$\|M_\delta(T)\|_{L^p}^p \lesssim \int_0^T \|a_\delta(s)\|_{L^p}^p (\|\alpha^\delta(s)\|_{L^3}^2 + \|\beta^\delta(s)\|_{H^{\frac{3}{2}}}^2) ds + M_\delta(T)^{p-1}\|F\|_{L^1L^p}.$$

Viewing the above equation as an algebraic inequality between positive numbers, we get

$$\|M_\delta(T)\|_{L^p} \lesssim \left(\int_0^T \|a_\delta(s)\|_{L^p}^p (\|\alpha^\delta(s)\|_{L^3}^2 + \|\beta^\delta(s)\|_{H^{\frac{3}{2}}}^2) ds\right)^{\frac{1}{p}} + \|F\|_{L^1L^p}.$$

Taking again the p-th power and owing to the inequality $(a+b)^p \lesssim a^p + b^p$, we have

$$\|M_\delta(T)\|_{L^p}^p \lesssim \left(\int_0^T \|a_\delta(s)\|_{L^p}^p (\|\alpha^\delta(s)\|_{L^3}^2 + \|\beta^\delta(s)\|_{H^{\frac{3}{2}}}^2) ds\right) + \|F\|_{L^1L^p}^p.$$

Finally, since $\|a_\delta(T)\|_{L^p} \le M_\delta(T)$ for all $T > 0$, Grönwall's inequality entails that, for some constant $C > 0$,

$$\|a_\delta(T)\|_{L^p} \le C\|F\|_{L^1L^p} \exp\left(C\int_0^T \|\alpha(s)\|_{L^3}^2 + \|\beta(s)\|_{H^{\frac{3}{2}}}^2 ds\right).$$

Having this bound and its analogue for the exponent $p = \frac{6}{5}$, thanks to the assumptions we did on F, we get a solution of our problem in both the energy spaces associated to $L^{\frac{6}{5}}$ and L^p. We conclude that this new solution is actually equal to a thanks to Lemma 9. □

Lemma 11. *Let v be a fixed, divergence free vector field in L^2H^1. Let a be a $L^\infty L^{\frac{6}{5}} \cap L^2W^{1,\frac{6}{5}}$ solution of the linear system*

$$\begin{cases} \partial_t a + \nabla\cdot(av) - \Delta a = \alpha a + \displaystyle\sum_{i,j=1,2} \varepsilon_{i,j}(\partial_j\beta_i)A_{3,i}^3 a + F_1 + F_2 \\ a(0) = 0, \end{cases}$$

with $\varepsilon_{i,j} \in \{0,1\}$ for any $1 \le i,j \le 2$. We assume that α lies in L^2L^3 and that all the β_i's are in $L^2H^{\frac{3}{2}}$. The exterior forces F_1 and F_2 belong respectively to $L^1L^{\frac{3}{2}} \cap L^1L^{\frac{6}{5}}$ and $L^{\frac{4}{3}}L^{\frac{6}{5}} \cap L^1L^{\frac{6}{5}}$. Then a is actually in $L^\infty L^{\frac{3}{2}} \cap L^2W^{1,\frac{3}{2}}$.

Sketch of proof. We essentially have to repeat the proof of Lemma 10, apart from estimating the term coming from F_2. Keeping the same notations as in the last proof, we have

$$\int_0^t \int_{\mathbb{R}^3} a_\delta(s,x)^{\frac{1}{2}} F^\delta(s,x) dx ds \le \int_0^t \|F^\delta(s)\|_{L^{\frac{6}{5}}} \|a_\delta(s)^{\frac{1}{2}}\|_{L^6} ds.$$

Using the identity $\|a_\delta(s)^{\frac{1}{2}}\|_{L^6} = \|a_\delta(s)^{\frac{3}{4}}\|_{L^4}^{\frac{2}{3}}$ and the Sobolev embedding $H^{\frac{3}{4}} \hookrightarrow L^4$, we get

$$\int_0^t \|F^\delta(s)\|_{L^{\frac{6}{5}}} \|a_\delta(s)^{\frac{1}{2}}\|_{L^6} ds \lesssim \int_0^t \|F^\delta(s)\|_{L^{\frac{6}{5}}} \|a_\delta(s)^{\frac{3}{4}}\|_{L^2}^{\frac{1}{6}} \||a(s)|^{-\frac{1}{4}}\nabla a(s)\|_{L^2}^{\frac{1}{2}} ds.$$

Now, Young inequality gives us, for some constant $C > 0$,

$$\int_0^t \|F^\delta(s)\|_{L^{\frac{6}{5}}} \|a_\delta(s)^{\frac{3}{4}}\|_{L^2}^{\frac{1}{6}} \||a(s)|^{-\frac{1}{4}}\nabla a(s)\|_{L^2}^{\frac{1}{2}} ds \le \frac{1}{10}\int_0^t \||a(s)|^{-\frac{1}{4}}\nabla a(s)\|_{L^2}^2 ds$$
$$+ \int_0^t \|a_\delta(s)^{\frac{3}{4}}\|_{L^2}^2 \|F^\delta(s)\|_{L^{\frac{6}{5}}}^{\frac{4}{3}} ds + C\int_0^t \|F^\delta(s)\|_{L^{\frac{6}{5}}}^{\frac{4}{3}} ds.$$

Plugging this final bound in the energy estimate performed in $L^{\frac{3}{2}}$, the rest of the proof is the same as for Lemma 10. □

Lemma 12. *Let v be a fixed, divergence free vector field in L^2H^1. Let A be a matrix-valued function in L^2L^3. Let K be a matrix whose coefficients are homogeneous, isotropic Fourier multipliers of order 0. Let F_1 be a fixed function in $L^1L^{\frac{3}{2}} \cap L^1L^{\frac{6}{5}}$ and F_2 be fixed in $L^{\frac{4}{3}}L^{\frac{6}{5}} \cap L^1L^{\frac{6}{5}}$. Let a be a solution in $(L^2L^2)^2$ of the equation*

$$\begin{cases} \partial_t a + \nabla\cdot(a\otimes v) - \Delta a = AKa + F_1 + F_2 \\ a(0) = 0. \end{cases}$$

Then a is actually in $L^\infty L^{\frac{3}{2}} \cap L^2W^{1,\frac{3}{2}}$.

Proof. This lemma essentially combines the proofs of Lemmas 6, 10 and 11, so we shall not repeat them. □

Lemma 13. *Let v_0 be a divergence free vector field in $L^{\frac{3}{2}} \cap L^2$. Then any Leray solution of the Navier–Stokes system*

$$\begin{cases} \partial_t v + \nabla\cdot(v\otimes v) - \Delta v = -\nabla p \\ div\ v = 0 \\ v(0) = v_0 \end{cases}$$

belongs, in addition to the classical energy space $L^\infty L^2 \cap L^2H^1$, to $L^\infty L^{\frac{3}{2}} \cap L^2W^{1,\frac{3}{2}}$.

Proof. Let v be a Leray solution of the Navier–Stokes system, which exists by classical approximation arguments. Then, letting $F := -\mathbb{P}\nabla\cdot(v\otimes v) = -\mathbb{P}(v\cdot\nabla v)$ where $\mathbb{P}$ denotes the Leray projection on divergence free vector fields, v solves the heat equation

$$\begin{cases} \partial_t v - \Delta v = F \\ v(0) = v_0. \end{cases}$$

That F belongs to $L^1L^{\frac{3}{2}}$ is easily obtained by the continuity of $\mathbb{P}$ on $L^{\frac{3}{2}}$. The result follows from an energy estimate in $L^{\frac{3}{2}}$. □

5. Case of the torus

Let us now state the first main theorem of this paper.

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Theorem 4. *Let u be a Leray solution of the Navier–Stokes equations set in $\mathbb{R}_+ \times \mathbb{T}^3$*

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) - \Delta u = -\nabla p \\ u(0) = u_0 \end{cases}$$

with initial data u_0 in $L^2(\mathbb{T}^3)$. Assume the existence of a time interval $]T_1, T_2[$ such that its third component u^3 satisfies

$$u^3 \in L^2(]T_1, T_2[, W^{2,\frac{3}{2}}(\mathbb{T}^3)).$$

Then u is actually smooth in space on $]T_1, T_2[\times \mathbb{T}^3$.

Let χ, φ be smooths cutoffs in time, localized inside $]T_1, T_2[$. Let ω be the third component of $\Omega := \operatorname{rot} v$. Denote $\chi\omega$ by ω'. The equation satisfied by ω' writes

$$\partial_t \omega' + \nabla \cdot (\omega' u) - \Delta \omega' = \chi \Omega \cdot \nabla u^3 + \omega \partial_t \chi.$$

Denote $F := \chi \Omega \cdot \nabla u^3 + \omega \partial_t \chi$. As u is a Leray solution of the Navier–Stokes equations, we know that Ω belongs to $L^2 L^2$. Thus, ω' also lies in $L^2 L^2$. On the other hand, the assumption made on u^3 tells us in particular that $\Omega \cdot \nabla u^3$ belongs to $L^1 L^{\frac{6}{5}}$. That $\omega \partial_t \chi$ also belongs to $L^1 L^{\frac{6}{5}}$ follows directly from the compactness of $\mathbb{T}^3$.

We are now in position to apply Lemma 6, which tells us that ω' is actually in $L^\infty L^{\frac{6}{5}} \cap L^2 W^{1,\frac{6}{5}}$. Let us now expand the quantity $\Omega \cdot \nabla u^3$ in terms of ω and u^3. We have, after some simplifications,

$$\Omega \cdot \nabla u^3 = \partial_3 u^3 \omega + \partial_2 u^3 \partial_3 u^1 - \partial_1 u^3 \partial_3 u^2.$$

Performing a div-curl decomposition of u^1 and u^2 in terms of $\partial_3 u^3$ and ω, we have

$$\begin{aligned} \Omega \cdot \nabla u^3 &= \partial_3 u^3 \omega + \partial_2 u^3 (-A^3_{1,3} \partial_3 u^3 - A^3_{2,3} \omega) - \partial_1 u^3 (-A^3_{2,3} \partial_3 u^3 + A^3_{1,3} \omega) \\ &= \partial_3 u^3 \omega + \mathcal{A}(\omega, u^3) + \mathcal{B}(u^3, u^3), \end{aligned}$$

where we defined as shorthands the operators

$$\begin{aligned} \mathcal{A}(\omega, u^3) &:= -\partial_2 u^3 A^3_{2,3} \omega - \partial_1 u^3 A^3_{1,3} \omega \\ \mathcal{B}(u^3, u^3) &:= -\partial_2 u^3 A^3_{1,3} \partial_3 u^3 + \partial_1 u^3 A^3_{2,3} \partial_3 u^3. \end{aligned}$$

Notice that the div-curl decomposition forces the appearance of weakly anisotropic operators acting either on ω or u^3. Assume from now on that the condition

$$\operatorname{supp} \chi \subset \{\varphi \equiv 1\}$$

holds. Under this condition, the equation on ω' then reads

$$\begin{aligned} \partial_t \omega' + \nabla \cdot (\omega' u) - \Delta \omega' &= \chi \omega \partial_3 u^3 + \chi \mathcal{A}(\omega, u^3) + \chi \mathcal{B}(u^3, u^3) + \omega \partial_t \chi \\ &= \omega' \partial_3 u^3 + \mathcal{A}(\omega', \varphi u^3) + \mathcal{B}(\chi u^3, \varphi u^3) + \omega \partial_t \chi, \end{aligned}$$

because the cutoffs χ and φ act only on time.

It follows from the assumptions on u^3 that $\mathcal{B}(\chi u^3, \varphi u^3)$ belongs to $L^1 L^{\frac{3}{2}}$. Moreover, $\omega \partial_t \chi$ also belongs to $L^1 L^{\frac{3}{2}}$.

By Lemma 10, ω' is actually in $L^\infty L^{\frac{3}{2}} \cap L^2 W^{1,\frac{3}{2}}$.

Let us now write the system of equations satisfied by the other components of the vorticity, which we respectively denote by ω_1 and ω_2. We have

$$\begin{cases} \partial_t \omega_1 + \nabla \cdot (\omega_1 u) - \Delta \omega_1 = \partial_3 u^1 \partial_1 u^2 - \partial_2 u^1 \partial_1 u^3 \\ \partial_t \omega_2 + \nabla \cdot (\omega_2 u) - \Delta \omega_2 = \partial_1 u^2 \partial_2 u^3 - \partial_3 u^2 \partial_2 u^1. \end{cases}$$

We now perform a div-curl decomposition of u^1 with respect to the second variable. That is, we write that

$$u^1 = \partial_3 \Delta^{-1}_{(1,3)} \omega_2 - \partial_1 \Delta^{-1}_{(1,3)} \partial_2 u^2.$$

In turn, we have

$$\begin{aligned} \partial_3 u^1 &= \partial_3^2 \Delta^{-1}_{(1,3)} \omega_2 - \partial_3 \partial_1 \Delta^{-1}_{(1,3)} \partial_2 u^2 \\ &= A^2_{3,3} \omega_2 - A^2_{1,3} \partial_2 u^2. \end{aligned}$$

What we wish to emphasize is that $\partial_3 u^1$ may be expressed as an order zero *isotropic* Fourier multiplier applied to ω_2 and $\partial_2 u^2$. The same reasoning applies to $\partial_3 u^2$, which may decomposed in terms of ω_1 et $\partial_1 u^1$. The fact that there is no (weakly) anisotropic operator here is a great simplification compared to the study of ω_3, for which such a complication was unavoidable. The system on (ω_1, ω_2) may be recast in the following form:

$$\begin{cases} \partial_t \omega_1 + \nabla \cdot (\omega_1 u) - \Delta \omega_1 = (A^2_{3,3} \omega_2 - A^2_{1,3} \partial_2 u^2) \partial_1 u^2 - \partial_2 u^1 \partial_1 u^3 \\ \partial_t \omega_2 + \nabla \cdot (\omega_2 u) - \Delta \omega_2 = \partial_1 u^2 \partial_2 u^3 + (A^1_{3,3} \omega_1 + A^1_{2,3} \partial_1 u^1) \partial_2 u^1. \end{cases}$$

Informally, the above system behaves roughly like its simplified version

$$\begin{cases} \partial_t \omega_1 + \nabla \cdot (\omega_1 u) - \Delta \omega_1 = (\omega_2 - \partial_2 u^2) \partial_1 u^2 - \partial_2 u^1 \partial_1 u^3 \\ \partial_t \omega_2 + \nabla \cdot (\omega_2 u) - \Delta \omega_2 = \partial_1 u^2 \partial_2 u^3 + (\omega_1 + \partial_1 u^1) \partial_2 u^1, \end{cases}$$

which is much simpler to understand and shall make the upcoming computations clearer. Let us denote, as we did for $\omega = \omega_3$, $\omega_1' := \chi \omega_1$ and $\omega_2' := \chi \omega_2$. Applying the time cutoff χ to the system on (ω_1, ω_2), we get

$$\begin{cases} \partial_t \omega_1' + \nabla \cdot (\omega_1' u) - \Delta \omega_1' = \varphi \partial_1 u^2 A^2_{3,3} \omega_2' - \varphi \partial_1 u^2 A^2_{1,3} (\chi \partial_2 u^2) - (\chi \partial_2 u^1)(\varphi \partial_1 u^3) + \omega_1 \partial_t \chi \\ \partial_t \omega_2' + \nabla \cdot (\omega_2' u) - \Delta \omega_2' = \varphi \partial_2 u^1 A^1_{3,3} \omega_1' + \varphi \partial_2 u^1 A^1_{2,3} (\chi \partial_1 u^1) + (\chi \partial_1 u^2)(\varphi \partial_2 u^3) + \omega_2 \partial_t \chi. \end{cases}$$

Finally, applying the same decomposition to u^1 and u^2, we have four equations of the type

$$\partial_1 u^1 = -A^3_{1,1} \omega_3 - A^3_{1,2} \partial_3 u^3,$$

which allow us to control, for $1 \leq i, j \leq 2$, $\partial_i u^j$ in $L^\infty L^{\frac{3}{2}} \cap L^2 W^{1,\frac{6}{5}}$ in terms of ω_3 and $\partial_3 u^3$ in the same space. Thus, what we have gained through the regularity enhancement on ω_3 is the control of four components of the Jacobian of u, in addition to the three provided by the assumption on u^3. For this reason, the system we have on (ω_1, ω_2) may be viewed as an affine and isotropic one with all exterior forces in scaling invariant spaces. For instance, $\varphi \partial_2 u^1$ belongs to $L^2 L^3$, while the exterior forces lie in $L^1 L^{\frac{3}{2}}$. Lemma 8 now implies that both ω_1' and ω_2' are in $L^\infty L^{\frac{3}{2}} \cap L^2 W^{1,\frac{3}{2}}$.

We now have proven that the whole vorticity Ω belongs to $L^4 L^2$ by Sobolev embeddings. In turn, it implies that the whole velocity field belongs to $L^4 H^1$. The main theorem then follows from the application of the usual Serrin criterion.

6. Local case in $\mathbb{R}^3$

We state the second main theorem of this paper.

Theorem 5. *Let u be a Leray solution of the Navier–Stokes equations set in $\mathbb{R}_+ \times \mathbb{R}^3$*

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) - \Delta u = -\nabla p \\ u(0) = u_0 \end{cases}$$

with initial data u_0 in $L^2(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3)$. Assume the existence of a time interval $]T_1, T_2[$ and a spatial domain $D \Subset \mathbb{R}^3$ of compact closure such that its third component u^3 satisfies

$$u^3 \in L^2(]T_1, T_2[, W^{2,\frac{3}{2}}(D)).$$

Then, on $]T_1, T_2[\times D$, u is actually smooth in space.

Let us describe in a few words our strategy for this case. Compared to the torus, there are two main differences to notice. First, since the assumption on u^3 was made on the whole space, the cutoffs acted only in time. The difference between the original Navier–Stokes equation and its truncated version was thus only visible in one term, rendering our strategy easier to apply. On the other hand, since the torus has finite measure, the Lebesgue spaces form a decreasing family of spaces. This fact allowed us to lose some integrability when we wanted to embed different forcing terms in the same space. This last difference will become visible when dealing with commutators between Fourier multipliers and the cutoff functions, thus lengthening a little bit the proof, compared to the torus case. For that technical reason, we added an assumption on the initial data which was trivially true in the torus case, thanks to the aforementioned embedding of Lebesgue spaces.

Let χ, φ be smooths cutoffs in space and time, localized inside $]T_1, T_2[\times D$. Let ω be the third component of $\Omega := \operatorname{rot} v$. Denote $\chi\omega$ by ω'. The equation satisfied by ω' writes

$$\partial_t \omega' + \nabla \cdot (\omega' u) - \Delta \omega' = \chi \Omega \cdot \nabla u^3 + \mathcal{C}(\omega, \chi),$$

where $\mathcal{C}(\omega, \chi)$ stands for all the cutoff terms. Namely, we have

$$\mathcal{C}(\omega, \chi) := \omega \partial_t \chi + \omega u \cdot \nabla \chi - \omega \Delta \chi - 2\nabla \omega \cdot \nabla \chi.$$

As χ is smooth and has compact support, we claim that $\mathcal{C}(\omega, \chi)$ belongs to $L^1 L^{\frac{3}{2}} + L^2 H^{-1}$. Because χ has compact support in space, the terms in $L^1 L^{\frac{3}{2}}$ also lie in $L^1 L^{\frac{6}{5}}$. Finally, the quantity $\chi \Omega \nabla u^3$ clearly belongs to $L^1 L^{\frac{6}{5}}$. Let now $\omega'_{(1)}$ be the unique solution in $L^\infty L^{\frac{6}{5}} \cap L^2 W^{1,\frac{6}{5}}$ of the equation

$$\partial_t \omega'_{(1)} + \nabla \cdot (\omega'_{(1)} u) - \Delta \omega'_{(1)} = \chi \Omega \cdot \nabla u^3 + \omega \partial_t \chi + \omega u \cdot \nabla \chi - \omega \Delta \chi$$

with the initial condition $\omega'_{(1)}(0) = 0$. This solution exists thanks to the combination of Lemma 1 with a regularization procedure for the velocity field u; it is unique thanks to Lemma 2, since the vector field u is in $L^2 H^1$. Similarly, let $\omega'_{(2)}$ be the unique solution in $L^\infty L^2 \cap L^2 H^1$ of

$$\partial_t \omega'_{(2)} + \nabla \cdot (\omega'_{(2)} u) - \Delta \omega'_{(2)} = -2\nabla \omega \cdot \nabla \chi$$

with the initial condition $\omega'_{(2)}(0) = 0$. Let $\tilde{\omega}' := \omega'_{(1)} + \omega'_{(2)} - \omega'$. From the regularity we have on each term, $\tilde{\omega}'$ belongs to $L^2_{loc} L^2$ and satisfies

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$$\partial_t \tilde{\omega}' + \nabla \cdot (\tilde{\omega}' u) - \Delta \tilde{\omega}' = 0$$

along with the initial condition $\tilde{\omega}'(0) = 0$. Lemma 2 then implies that $\tilde{\omega}' \equiv 0$, from which it follows that

$$\omega' = \omega'_{(1)} + \omega'_{(2)}$$

By local embeddings of Lebesgue spaces, $\omega'_{(2)}$ also belongs to $L^\infty L^{\frac{6}{5}}_{loc} \cap L^2 W^{1,\frac{6}{5}}_{loc}$. On the other hand, it is rather trivial that $\omega'_{(1)}$ also belongs to $L^\infty L^{\frac{6}{5}}_{loc} \cap L^2 W^{1,\frac{6}{5}}_{loc}$. Now, since ω' has compact support in space, it follows that ω' belongs to the full space $L^\infty L^{\frac{6}{5}} \cap L^2 W^{1,\frac{6}{5}}$. In particular, the forcing term $\nabla \omega \cdot \nabla \chi$ is now an integrable vector field, instead of a mere $L^2 H^{-1}$ distribution. At this stage, because the reasoning is valid for *any* cutoff χ supported in $]T_1, T_2[\times D$, we have proved that the third component ω of the vorticity of u has the regularity

$$\omega \in L^\infty_{loc}(]T_1, T_2[, L^{\frac{6}{5}}_{loc}(D)) \cap L^2_{loc}(]T_1, T_2[, W^{1,\frac{6}{5}}_{loc}(D)).$$

In particular, such a statement allows us to improve the regularity of $\mathcal{C}(\omega, \chi)$ to $L^1 L^{\frac{3}{2}} + L^2 L^{\frac{6}{5}}$. Such a gain will be of utmost importance near the end of the proof. Expanding again the product $\Omega \cdot \nabla u^3$ in terms of ω and u^3 only, we have

$$\partial_t \omega' + \nabla \cdot (\omega' u) - \Delta \omega' = \chi \omega \partial_3 u^3 + \chi \mathcal{A}(\omega, u^3) + \chi \mathcal{B}(u^3, u^3) + \mathcal{C}(\omega, \chi).$$

From now on, we enforce the condition

$$\text{supp } \chi \subset \{\varphi \equiv 1\}.$$

Now, because the cutoff χ acts both in space and time, we have to carefully compute the associated commutators with the operators $\mathcal{A}$ and $\mathcal{B}$. First, let us notice that $\mathcal{A}$ is local in its variable u^3, which allows us to write that

$$\chi \mathcal{A}(\omega, u^3) = \chi \mathcal{A}(\omega, \varphi u^3).$$

On the other hand, for $i = 1, 2$,

$$\begin{aligned}
\chi A^3_{i,3} \omega &= \chi \partial_i \Delta^{-1}_{(1,2)} (\partial_3 \omega) \\
&= [\chi, \partial_i \Delta^{-1}_{(1,2)}](\partial_3 \omega) + \partial_i \Delta^{-1}_{(1,2)} (\chi \partial_3 \omega) \\
&= [\chi, \partial_i \Delta^{-1}_{(1,2)}](\partial_3 \omega) + A^3_{i,3}(\chi \omega) - \partial_i \Delta^{-1}_{(1,2)} (\omega \partial_3 \chi)
\end{aligned}$$

We now estimate the two remainder terms in $L^1 L^{\frac{3}{2}}$. By Sobolev embeddings in $\mathbb{R}^2$, we have, for $t > 0$ and $x_3 \in \mathbb{R}$,

$$\| \left(\partial_i \Delta^{-1}_{(1,2)} (\omega \partial_3 \chi) \right) (t, \cdot, x_3) \|_{L^6(\mathbb{R}^2)} \lesssim \| (\omega \partial_3 \chi)(t, \cdot, x_3) \|_{L^{\frac{3}{2}}(\mathbb{R}^2)}.$$

Thus,

$$\| \partial_i \Delta^{-1}_{(1,2)} (\omega \partial_3 \chi) \|_{L^2 L^{\frac{3}{2}} L^6} \lesssim \| \omega \partial_3 \chi \|_{L^2 L^{\frac{3}{2}}} \lesssim \| \omega \|_{L^2 L^2} \| \nabla \chi \|_{L^\infty L^6}.$$

The commutator is a little bit trickier. First, we write

$$\partial_3\omega = \partial_3(\partial_1 u^2 - \partial_2 u^1) = \partial_1(\partial_3 u^2) - \partial_2(\partial_3 u^1).$$

In order to continue the proof, we need a commutator lemma, which we state and prove below for the sake of completeness, despite its ordinary nature.

Lemma 14. *Let f be in $L^{\frac{3}{2}}(\mathbb{R}^2)$ and χ be a test function. The following commutator estimates hold:*

$$\|[\chi, \nabla\Delta^{-1}](\nabla f)\|_{L^6(\mathbb{R}^2)} \lesssim \|\nabla\chi\|_{L^\infty(\mathbb{R}^2)}\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}$$

and

$$\|[\chi, \nabla^2\Delta^{-1}](f)\|_{L^6(\mathbb{R}^2)} \lesssim \|\nabla\chi\|_{L^\infty(\mathbb{R}^2)}\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}.$$

Proof. We notice that the first estimate may be deduced from the second thanks to the identity

$$[\chi, \nabla\Delta^{-1}](\nabla f) = [\chi, \nabla^2\Delta^{-1}](f) + \nabla\Delta^{-1}(f\nabla\chi).$$

Since the operator $\nabla\Delta^{-1}$ is continuous from $L^{\frac{3}{2}}(\mathbb{R}^2)$ to $L^6(\mathbb{R}^2)$, we get

$$\|\nabla\Delta^{-1}(f\nabla\chi)\|_{L^6(\mathbb{R}^2)} \lesssim \|f\nabla\chi\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \lesssim \|f\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}\|\nabla\chi\|_{L^\infty(\mathbb{R}^2)}$$

It only remains to study the second commutator, which we denote by C_χ. There exist numerical constants c_1, c_2 such that, for almost every $x \in \mathbb{R}^2$,

$$C_\chi(x) = \int_{\mathbb{R}^2} \left(c_1\frac{(x-y)\otimes(x-y)}{|x-y|^4} + \frac{c_2}{|x-y|^2}I_2\right)(\chi(x)-\chi(y))f(y)dy.$$

This yields

$$|C_\chi(x)| \lesssim \|\nabla\chi\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} \frac{|f(y)|}{|x-y|}dy = \|\nabla\chi\|_{L^\infty(\mathbb{R}^2)}(|f| * |\cdot|^{-1})(x).$$

Applying the Hardy–Littlewood–Sobolev inequality to f, we get

$$\|C_\chi\|_{L^6(\mathbb{R}^2)} \lesssim \|\nabla\chi\|_{L^\infty(\mathbb{R}^2)}\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}$$

as we wanted. □

Thanks to Lemma 14, we have the estimate

$$\|[\chi, \partial_i\Delta^{-1}_{(1,2)}](\partial_1(\partial_3 u^2))\|_{L^6(\mathbb{R}^2)} \lesssim \|\nabla\chi\|_{L^\infty}\|\partial_3 u^2\|_{L^{\frac{3}{2}}(\mathbb{R}^2)},$$

which translates into

$$\|[\chi, \partial_i\Delta^{-1}_{(1,2)}](\partial_1(\partial_3 u^2))\|_{L^2L^{\frac{3}{2}}L^6} \lesssim \|\nabla\chi\|_{L^\infty}\|\partial_3 u^2\|_{L^2L^{\frac{3}{2}}}.$$

From Lemma 13 applied to u, we deduce that, in particular, $\partial_3 u^2$ belongs to $L^2L^{\frac{3}{2}}$. Moreover, we may bound $\|\partial_3 u^2\|_{L^2L^{\frac{3}{2}}}$ by a quantity depending only on u_0 through its L^2 and $L^{\frac{3}{2}}$ norms. Gathering these estimates, we may write

$$\chi\mathcal{A}(\omega,\varphi u^3)=\mathcal{A}(\chi\omega,\varphi u^3)+\mathcal{R}(\mathcal{A}),$$

with the remainder $\mathcal{R}(\mathcal{A})$ bounded in $L^1L^{\frac{3}{2}}$ only in terms of the initial data u_0, the cutoff χ and u^3. In particular, it may be regarded as an exterior force independant of ω' in the sequel and scaling invariant. The same reasoning applies to $\mathcal{B}$: we have

$$\chi\mathcal{B}(u^3,\varphi u^3)=\mathcal{B}(\chi u^3,\varphi u^3)+\mathcal{R}(\mathcal{B}),$$

with $\mathcal{R}(\mathcal{B})$ bounded in $L^1L^{\frac{3}{2}}$ only in terms of χ and u^3. Finally, the equation on ω' has been rewritten as

$$\partial_t\omega'+\nabla\cdot(\omega' u)-\Delta\omega'=\omega'\partial_3u^3+\mathcal{A}(\omega',\varphi u^3)+\mathcal{B}(\chi u^3,\varphi u^3)+\mathcal{C}(\omega,\chi)+\mathcal{R}(\mathcal{A})+\mathcal{R}(\mathcal{B}).$$

Applying Lemma 11, we deduce that the truncated vorticity ω' is actually in $L^\infty L^{\frac{3}{2}}\cap L^2W^{1,\frac{3}{2}}$. Again, thanks to the div-curl decomposition, it follows that space–time truncations of ∂_iu^j are controlled in the same space in terms of ω' and u^3, for $1\le i,j\le 2$. We now turn to the other components of the vorticity, namely ω_1 and ω_2. Truncating the equations and using the div-curl decomposition, we have

$$\begin{cases}\partial_t\omega_1'+\nabla\cdot(\omega_1'u)-\Delta\omega_1'=\chi(A^2_{3,3}\omega_2-A^2_{1,3}\partial_2u^2)\partial_1u^2-\chi\partial_2u^1\partial_1u^3+\mathcal{C}(\omega_1,\chi)\\ \partial_t\omega_2'+\nabla\cdot(\omega_2'u)-\Delta\omega_2'=\chi\partial_1u^2\partial_2u^3+\chi(A^1_{3,3}\omega_1+A^1_{2,3}\partial_1u^1)\partial_2u^1+\mathcal{C}(\omega_2,\chi).\end{cases}$$

Let us now write and estimate the necessary commutators. By Lemma 14, we have, when k is neither i nor j,

$$\|[\chi,A^k_{i,j}](\omega_2)\|_{L^6(\mathbb{R}^2)}\lesssim\|\nabla\chi\|_{L^\infty}\|\omega_2\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}.$$

Thus,

$$\|[\chi,A^k_{i,j}](\omega_2)\|_{L^2L^{\frac{3}{2}}L^6}\lesssim\|\nabla\chi\|_{L^\infty}\|\omega_2\|_{L^2L^{\frac{3}{2}}}.$$

On the other hand, by a trace theorem, we have, for a in $W^{1,\frac{3}{2}}(\mathbb{R}^3)$,

$$\|a\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}^2))}\lesssim\|a\|_{W^{1,\frac{3}{2}}(\mathbb{R}^3)}.$$

These two estimates together entail that, for $1\le i,j\le 2$,

$$\|\partial_i(\varphi u^j)[\chi,A^k_{i,j}](\omega_2)\|_{L^1L^{\frac{3}{2}}}\lesssim\|\nabla\chi\|_{L^\infty}\|\omega_2\|_{L^2L^{\frac{3}{2}}}\|\partial_i(\varphi u^j)\|_{W^{1,\frac{3}{2}}}.$$

The system on (ω_1',ω_2') may be recast as

$$\begin{cases}\partial_t\omega_1'+\nabla\cdot(\omega_1'u)-\Delta\omega_1'=(A^2_{3,3}\omega_2'-A^2_{1,3}\partial_2(\chi u^2))\partial_1(\varphi u^2)-\partial_2(\chi u^1)\partial_1(\varphi u^3)+\mathcal{C}(\omega_1,\chi)+\mathcal{R}^1\\ \partial_t\omega_2'+\nabla\cdot(\omega_2'u)-\Delta\omega_2'=\partial_1(\chi u^2)\partial_2(\varphi u^3)+(A^1_{3,3}\omega_1'+A^1_{2,3}\partial_1(\chi u^1))\partial_2(\varphi u^1)+\mathcal{C}(\omega_2,\chi)+\mathcal{R}^2,\end{cases}$$

where the remainders $\mathcal{R}^{1,2}$ contain, among other terms, the commutators we just estimated. The important fact is the boundedness of $\mathcal{R}^{1,2}$ in $L^1L^{\frac{3}{2}}$. Because χ has compact support in time, the term $-2\nabla\omega\cdot\nabla\chi$ is in $L^{\frac{4}{3}}L^{\frac{6}{5}}$. Applying Lemma 12, it follows that both ω_1' and ω_2' belong to $L^\infty L^{\frac{3}{2}}\cap L^2W^{1,\frac{3}{2}}$. The conclusion of the theorem now follows from the standard Serrin criterion.

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